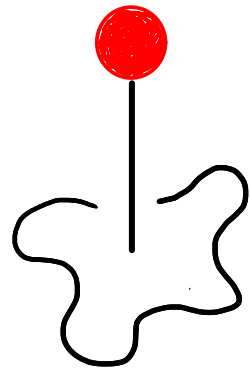




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Here we are

Topological Spaces

$$\begin{array}{ccc} f: X & \longrightarrow & Y \\ \downarrow & & \downarrow \\ f_{\mathbb{Q}}: X_{\mathbb{Q}} & \longrightarrow & Y_{\mathbb{Q}} \end{array}$$

Talk 1

$$X \rightsquigarrow \text{APL}(X)$$

Talk 4

For spaces that satisfy some conditions



M_X has information about the RHT of the space

- $\pi_*(X) \otimes \mathbb{Q} = \pi_*(X_{\mathbb{Q}})$
 - Rational homotopy type of a space
- $\Downarrow$
- weak homotopy type of its rationalization

CDG Algebras

$$\begin{array}{ccc} (A, d) & \xrightarrow{g} & (B, d) \\ \uparrow \varphi_A & & \uparrow \varphi_B \\ (M_A, d) & \xrightarrow{i?} & (M_B, d) \end{array}$$

Talk 2 Talk 3

- Is the model unique?
- How the models are related?
- weak equivalence class of CDGA's

Talk 5

Fibrations and the fundamental theorem of RHT

$$C_{\text{TOP}} \rightsquigarrow C_{\text{CDGA}}$$

Both categories are objects of a category called "model category"

Outline

- 1) (Re) definition of Sullivan algebras
- 2) Examples of minimal models
- 3) the lifting problem

(Relative) Sullivan algebras

Def: A relative Sullivan algebra is a commutative cochain algebra

$$(B \otimes \Lambda V, d)$$

with the following properties:

- the "base algebra" $(B, d) = (B \otimes 1, d)$ is a subcochain algebra and connected, i.e., $H^0(B, d) = \mathbb{Q}$
- the algebra ΛV is connected and built upon a graded vector space $1 \otimes V = V = \{V^p\}_{p \geq 1}$

(Relative) Sullivan algebras

- there exists a filtration

$$V = \bigcup_{k=0}^{\infty} V(k)$$

of graded subspaces

$$V(0) \subseteq V(1) \subseteq V(2) \subseteq \dots \subseteq V$$

such that

$$d: V(0) \longrightarrow B$$

and

$$d: V(k) \longrightarrow B \otimes \Lambda(V(k-1)), \text{ for } k \geq 1$$

(Relative) Sullivan algebras

A Sullivan algebra is a relative Sullivan algebra
with $(B, d) = (\mathbb{Q}, 0)$

A relative Sullivan algebra is minimal if

$$\text{im } d \subseteq B^{\geq 1} \otimes \Lambda V + B \otimes \Lambda^{\geq 2} V$$

For a Sullivan algebra, this is equivalent to

$$\text{im } d \subseteq \Lambda^{\geq 2} V$$

(Relative) Sullivan algebras: Examples

1) A CDGA that is not a Sullivan algebra: [Manero & Marco 2020]

Let A be a CDGA generated by $\langle e_1, \dots, e_7 \rangle$

where $|e_i| = 1$, for $i \leq 6$

$$|e_7| = 2$$

and the differential is given by:

$$\begin{aligned} d(e_1) &= -e_1 \wedge e_6 \\ d(e_2) &= -e_2 \wedge e_6 \\ d(e_3) &= -e_3 \wedge e_6 \end{aligned}$$

$$d(e_4) = -e_5 \wedge e_6$$

$$d(e_5) = d(e_6) = d(e_7) = 0$$

It is not a Sullivan algebra because the differentials of e_1, e_2 and e_3 are given as a product of themselves.

(Relative) Sullivan algebras: Examples

2) A Sullivan algebra (not relative
not minimal)

$$(\Lambda(y, t), d) \quad \begin{array}{ll} |y| = 3 & dy = 0 \\ |t| = 2 & dt = y \end{array}$$

It is a Sullivan algebra
but it is not minimal
because it is not satisfied
the condition $\text{Im } d \subseteq \Lambda^{\geq 2} V$

3) A minimal Sullivan algebra

$$(\Lambda_2(x, y, z), d) \quad \begin{array}{l} |x| = |y| = |z| = 1 \\ dx = dy = 0 \\ dz = xy \end{array}$$

(Relative) Sullivan algebras: Examples

4) A relative Sullivan algebra

Let (B, d_B) be the CDGA $(\Lambda(y, t), d)$ with $|y|=2$ $d(y)=0$
 $|t|=3$ $d(t)=y^2$

and let (A, d_A) be the CDGA $(\Lambda(x), d)$ with $|x|=1$
 $d(x)=0$

Using (B, d_B) as a "base algebra", we can construct the

relative Sullivan algebra $(B \otimes A, D) = (\Lambda(y, t, x), D)$

with $D|_B = d_B$

$$D(x) = y$$

$$H^3(B \otimes A) = \mathbb{Q} \langle [xy - t] \rangle$$

Note:

this algebra comes from the Hopf fibration

$$S^1 \rightarrow S^3 \rightarrow S^2$$
$$(A, d_A) \leftarrow (B \otimes A, D) \leftarrow (B, d_B)$$

The algorithm for the minimal model

Let (A, d_A) be a connected ($H^0(A, d_A) = \mathbb{Q}$) commutative cochain algebra

$$\varphi: (M, d_M) \longrightarrow (A, d_A)$$

Step 1) Find the first $i \geq 1$ for which $H^i(A, d_A) \neq 0$

Add generators x_j to M^i ($|x_j| = i$) in such a way that $\varphi_*^i: H^i(M, d_M) \longrightarrow H^i(A, d_A)$ is an isomorphism.

The algorithm for the minimal model

step 2)

if $\varphi_{\star}^{i+1}: H^{i+1}(M, d_M) \rightarrow H^{i+1}(A, d_A)$ is injective

$\Rightarrow$ go back to step 1 and repeat the process for degree $i+1$

else ($\varphi_{\star}^{i+1}$ not injective)

Add to M^i generators y_j in such a way
you make $\varphi_{\star}^{i+1}$ injective.

Example: A contractible algebra

Def: A Sullivan algebra $(\Lambda V, d)$ is called contractible

$$\text{if } H^n(\Lambda V, d) = \begin{cases} \mathbb{Q} & \text{if } n=0 \\ 0 & \text{otherwise} \end{cases}$$

Let $(\Lambda V, d)$ be the algebra generated by $\langle t, dt \rangle$
with $|t|=1$

It is easy to see that $(\Lambda V, d)$ is contractible

and, for that reason, its minimal model is $M = \mathbb{Q}$.

Example

[Manera & Marco 2020]

Let (A, d_A) be the CDGA generated by $\langle e_1, \dots, e_7 \rangle$
with $|e_i| = 1$, $i \leq 6$, has no relations
 $|e_7| = 2$

and the differential is given by

$$d_A(e_1) = -e_1 \wedge e_6$$

$$d_A(e_2) = -e_2 \wedge e_6$$

$$d_A(e_3) = -e_3 \wedge e_6$$

$$d_A(e_4) = -e_5 \wedge e_6$$

$$d_A(e_5) = d_A(e_6) = d_A(e_7) = 0$$

the first cohomology groups are

$$H^1(A) = \mathbb{Q} \langle [e_5], [e_6] \rangle$$

$$H^2(A) = \mathbb{Q} \langle [e_4 \wedge e_5], [e_4 \wedge e_6], [e_7] \rangle$$

$$H^3(A) = \mathbb{Q} \langle [e_4 \wedge e_5 \wedge e_6], [e_5 \wedge e_7], [e_6 \wedge e_7] \rangle$$

Example

[Manero & Marco 2020]

So we start the algorithm for degree 1

$$\begin{array}{ccccc} e_6 & \xleftarrow{\varphi} & x'_0 & \xrightarrow{d_M} & 0 \\ e_5 & \xleftarrow{\quad} & x'_1 & \xrightarrow{\quad} & 0 \end{array}$$

At this point, $H^1(\varphi)$ is an isomorphism and $H^2(M)$ would be generated by the class $[x'_0 \wedge x'_1]$ but,

$$\varphi(x'_0 \wedge x'_1) = -e_5 \wedge e_6 \quad \text{which is trivial in cohomology} \\ (d(e_4) = -e_5 \wedge e_6)$$

So, we have to add a new generator in order to make this cohomology class trivial:

$$e_4 \xleftarrow{\varphi} y'_0 \xrightarrow{d_M} x'_0 \wedge x'_1$$

Example [Manero & Marco 2020]

Now we have $M = \langle x'_0, x'_1, y'_0 \rangle$ and we have to check again the new elements that have been created in higher degrees.

$$d_M(x'_0 \wedge y'_0) = d_M(x'_1 \wedge y'_0) = 0$$

$$\text{so } H_M^2 = \mathbb{Q} \langle [x'_0 \wedge y'_0], [x'_1 \wedge y'_0] \rangle$$

In that case, we have that

$$\left. \begin{array}{l} x'_0 \wedge y'_0 \longrightarrow e_6 \wedge e_4 \\ x'_1 \wedge y'_0 \longrightarrow e_5 \wedge e_4 \end{array} \right\} \text{ which are two representatives of } \\ \text{two elements of the basis of } H^2(A)$$

so the map φ_* is injective for degree 2

Example [Manero & Marco 2020]

$$H^1(A) = \mathbb{Q} \langle \cancel{[e_5]}, \cancel{[e_6]} \rangle$$

$$H^2(A) = \mathbb{Q} \langle \cancel{[e_4 \wedge e_5]}, \cancel{[e_4 \wedge e_6]}, \cancel{[e_7]} \rangle$$

$$H^3(A) = \mathbb{Q} \langle \cancel{[e_4 \wedge e_5 \wedge e_6]}, \cancel{[e_5 \wedge e_7]}, \cancel{[e_6 \wedge e_7]} \rangle$$

$$H^1(M) = \langle [x'_0], [x'_1] \rangle$$

$$H^2(M) = \langle [x'_0 \wedge y'_0], [x'_1 \wedge y'_0], [x_0^2] \rangle$$

$$H^3(M) = \langle [x'_0 \wedge x'_1 \wedge y'_0], [x'_0 \wedge x_0^2], [x'_1 \wedge x_0^2] \rangle$$

Now we continue creating the isomorphism in cohomology at degree 2. For that, we add to M a generator

$$e_7 \longleftarrow \varphi \quad x_0^2 \xrightarrow{d_M} 0$$

and we have an isomorphism at $H^2(\varphi)$

But also we have an isomorphism at degree 3 because

$$d(x'_0 \wedge x'_1 \wedge y'_0) = 0$$

$$d(x'_0 \wedge x_0^2) = 0$$

$$d(x'_1 \wedge x_0^2) = 0$$

$$\varphi(x'_0 \wedge x'_1 \wedge y'_0) \longrightarrow e_6 \wedge e_5 \wedge e_4$$

$$\varphi(x'_0 \wedge x_0^2) \longrightarrow e_6 \wedge e_7$$

$$\varphi(x'_1 \wedge x_0^2) \longrightarrow e_5 \wedge e_7$$

Example: the minimal model of $(H^*(S^n; \mathbb{Q}), d=0)$ for $n \geq 2$

$$H^q(S^n, \mathbb{Q}) = \begin{cases} \langle h^q \rangle & \text{if } q = \{0, n\} \\ 0 & \text{otherwise} \end{cases} \quad \text{with } (h^n)^2 = 0$$

First, consider 'n' odd:

$$h^n \xleftarrow{\varphi} x_0^n \xrightarrow{d_n} 0$$

as $|x_0^n|$ is odd, $(x_0^n)^2 = 0$ so we have

that $\boxed{M \cong H^*(S^n)}$

Example: the minimal model of $H^*(S^n)$

Consider now the case 'n' even

$$h^n \xleftarrow{\varphi} x_0^n \xrightarrow{d_M} 0$$

Now we have an isomorphism at degree 'n'
but because 'n' is even, we have

$$H^{2n}(M) = \mathbb{Q} \langle [x_0^n]^2 \rangle$$

$$\text{but } H^{2n}(S^n) = 0$$

So we need to add a generator of degree $2n-1$
in order to make this cohomology trivial

$$0 \xleftarrow{\varphi} y_0^{2n-1} \xrightarrow{d_M} (x_0^n)^2$$

$$\text{So } \boxed{M = \mathbb{Q} \langle x_0^n, y_0^{2n-1} \rangle}.$$

Example: A non-minimal model for $H^*(S^n)$ (n odd)

You can consider a non-minimal model for $H^*(S^n)$

$$\begin{array}{lll} \tilde{M} = \langle x, b, a \rangle & \text{where} & |x| = n \quad d(x) = 0 \\ \varphi(x) = [S^n] & & |b| = n+1 \quad d(b) = 1 \\ \varphi(a) = \varphi(b) = 0 & & |a| = n \quad d(a) = 0 \end{array}$$

In that case, you can split $\tilde{M}$ as a product of a minimal Sullivan algebra, $M = \langle x \rangle$, and a contractible Sullivan algebra, $\tilde{C} = \langle b, a \rangle$,

$$\tilde{M} \cong M \otimes \tilde{C}$$

this is why we need
minimality for
uniqueness

so that

$$\boxed{H^*(\tilde{M}) \cong H^*(M)}$$

Example: $H^*(S^2 \wedge S^2)$

$$H^2(S^2 \wedge S^2) = \mathbb{Q}\langle h_0^2, h_1^2 \rangle$$

So we start the algorithm

h_0^2	$\xleftarrow{\varphi}$	x_0^2	$\xrightarrow{d_H}$	0
h_1^2	$\xleftarrow{\quad}$	x_1^2	$\xrightarrow{\quad}$	0
0	$\xleftarrow{\quad}$	y_0^3	$\xrightarrow{\quad}$	$(x_0^2)^2$
0	$\xleftarrow{\quad}$	y_1^3	$\xrightarrow{\quad}$	$x_0^2 x_1^2$
0	$\xleftarrow{\quad}$	y_2^3	$\xrightarrow{\quad}$	$(x_1^2)^2$
0	$\xleftarrow{\quad}$	y_0^4	$\xrightarrow{\quad}$	$x_1^2 y_0^3 - x_0^2 y_1^3$
0	$\xleftarrow{\quad}$	y_1^4	$\xrightarrow{\quad}$	$x_1^2 y_1^3 - x_0^2 y_2^3$
		$\vdots$		

B) The lifting problem

$$\begin{array}{ccc} (B, d) & \xrightarrow{h} & (A, d) \\ i \downarrow & \nearrow \tilde{f} & \simeq \downarrow g \\ (B \otimes \Lambda V, d) & \xrightarrow{f} & (C, d) \end{array}$$

- The outer square commutes
- $(B \otimes \Lambda V, d)$ is a relative Sullivan algebra
- the morphism i is the inclusion of the base
- g is a quasi-isomorphism

Lemma (lifting lemma): the lifting problem has a solution $\tilde{f}$ up to homotopy relative to B , i.e. the morphism $\tilde{f}$ of CDGA's exists and makes the diagram commute up to homotopy relative to B

Proof

Step 1: Assume $g: (A, d) \xrightarrow{\cong} (C, d)$ is surjective and, inductively, we show strict commutativity.

$(B \otimes \Lambda V, d)$ is a relative Sullivan algebra, so

$$V(K) = V(K-1) \oplus V_K$$

Also, we have that $d: V_K \longrightarrow B \otimes \Lambda V(K-1)$

By induction, we shall now lift f to $B \otimes \Lambda V(K)$

(it suffices to construct a lift of f for the elements of V_K)

For $K=0$, we have $V(0)=0$ and a lift on $(B \otimes \Lambda V(0), d) = (B, d)$ is given by the morphism h .

Proof

Suppose now that the lift
has been constructed for $V(k-1)$.

$$\begin{array}{ccc} (B, d) & \xrightarrow{h} & (A, d) \\ i \downarrow & \nearrow \tilde{f} & \cong \downarrow g \\ (B \otimes \Lambda V, d) & \xrightarrow{f} & (C, d) \end{array}$$

Let $\sigma \in V_k$. By definition, $d\sigma \in B \otimes \Lambda V(k-1)$. thus

$$(g \circ \tilde{f})(d\sigma) = f(d\sigma) = d(f(\sigma))$$

Moreover, $d\tilde{f}(d\sigma) = \tilde{f}(d^2\sigma) = 0 \implies \tilde{f}(d\sigma)$ defines a
cohomology class
with the property

$$H(g)([\tilde{f}(d\sigma)]) = [d(f(\sigma))] = 0$$

Since g is a quasi-isomorphism, we conclude that there is
an element $w \in A$ with $dw = \tilde{f}(d\sigma)$

this element is a potential candidate for $\tilde{f}(\sigma)$

but we still need to correct it slightly.

Proof

Now consider the term

$g(w) - f(v)$ (if $\neq 0$, the diagram doesn't commute)

and compute

$$d(g(w) - f(v)) = g(dw) - f(dv) = (g \circ \tilde{f})(dv) - f(dv) = 0$$

thus, also $g(w) - f(v)$ defines a cohomology class.

Since g is a quasi-isomorphism, there is a closed element $w' \in A$ with $g(w') = g(w) - f(v) + dv'$ for some $v' \in C$.

Since g is surjective, there is an element $w'' \in A$ with $g(w'') = v'$ and with $g(dw'') = dv'$

$$\text{thus, } g(w' - dw'') = g(w) - f(v)$$

$$\begin{array}{ccc} (B, d) & \xrightarrow{h} & (A, d) \\ i \downarrow & \nearrow \tilde{f} & \downarrow g \\ (B \otimes \Lambda V, d) & \xrightarrow{f} & (C, d) \end{array}$$

Proof

Set

$$\tilde{f}(v) = w - w' + dw''$$

then the triangle commutes, since

$$g(\tilde{f}(v)) = g(w) - (g(w) - f(v) + dv') + dv' = f(v)$$

and also, $\tilde{f}$ commutes with differentials

$$d\tilde{f}(v) = dw - dw' + d^2w'' = dw = \tilde{f}(dv) \quad \text{since } w' \text{ is closed}$$

Consequently, $\tilde{f}$ is a morphism of differential graded algebras, making the diagram commute, i.e. a lift of f .

$$\begin{array}{ccc} (B, d) & \xrightarrow{h} & (A, d) \\ i \downarrow & \tilde{f} \nearrow & \downarrow g \\ (B \otimes \Lambda V, d) & \xrightarrow{f} & (C, d) \end{array}$$

Proof

Step 2: Consider now the general case when g is not surjective

(Idea: make it surjective)

Consider the commutative differential graded algebra $(E(C), \delta)$ defined as:

$$E(C) = \Lambda(C \oplus \delta C)$$

where δC is considered as a formal copy of C , shifted in degree by $+1$.

In other words, δ is a linear map of degree $+1$ injective on C , and trivial on δC .

$(E(C, d), \delta)$ is of the form $(X) \Lambda(w_i, dw_i)$
and it is contractible

$$\begin{array}{ccc} (B, d) & \xrightarrow{h} & (A, d) \\ i \downarrow & \nearrow \tilde{f} & \simeq \downarrow g \\ (B \otimes \Lambda V, d) & \xrightarrow{f} & (C, d) \end{array}$$

Proof

there exists a surjective morphism
of differential cochain algebras

$$\varphi: (E(C, d), \delta) \longrightarrow (C, d)$$

given by

$$\begin{array}{ccc} c & \longrightarrow & c \\ \delta c & \longrightarrow & dc \end{array}$$

$$\begin{array}{ccc} (B, d) & \xrightarrow{h} & (A, d) \\ i \downarrow & \nearrow \tilde{f} & \simeq \downarrow g \\ (B \otimes \Lambda V, d) & \xrightarrow{f} & (C, d) \end{array}$$

$$\left(\begin{array}{l} \text{Indeed} \\ d\varphi(c) = dc = \varphi(\delta c) \\ \text{for } c \in C \end{array} \right)$$

We may replace (A, d) by $(A, d) \otimes (E(C, d), \delta)$
and the morphism g by

$$\tilde{g}: (A, d) \otimes (E(C, d), \delta) \xrightarrow{g \cdot \varphi} (C, d)$$

which is surjective, since it is φ .

Also, we extend the morphism

$$h \rightarrow h \otimes 1$$

$$\begin{array}{ccc} (B, d) & \xrightarrow{h} & (A, d) \otimes (E(C, d), \delta) \\ i \downarrow & \nearrow \tilde{f}' & \simeq \downarrow \tilde{g} \\ (B \otimes \Lambda V, d) & \xrightarrow{f} & (C, d) \end{array}$$

$\tilde{g}$ is also a
quasi-isomorphism

so due to step 1,
the lifting problem
has a solution C up to homot. rel B

Important results (without proof)

Proposition

P.1.) If $f_0 \simeq f_1 : (\Lambda V, d) \longrightarrow (A, d)$, then $H(f_0) = H(f_1)$

Lemma (homotopy lifting lemma): Let $g : (A, d) \longrightarrow (C, d)$ be a quasi-isomorphism of commutative differential graded algebras and let $\tilde{f}_1, \tilde{f}_2 : (B \otimes \Lambda V, d) \longrightarrow (A, d)$ be two morphisms of CDGAs with a relative Sullivan algebra $(B \otimes \Lambda V, d)$.

| if $g \circ \tilde{f}_1 \simeq g \circ \tilde{f}_2$, then already $\tilde{f}_1 \simeq_B \tilde{f}_2$ |

$$\begin{array}{ccc} (B \otimes \Lambda V, d) & \begin{array}{c} \xrightarrow{\tilde{f}_1} \\ \xrightarrow{\tilde{f}_2} \end{array} & (A, d) \\ & & \downarrow \simeq g \\ & & (C, d) \end{array}$$

Important results (without proof)

Prop. Let $(\Delta V, d)$ and $(\Delta W, d)$ be minimal Sullivan algebras which are quasi-isomorphic via

$$g: (\Delta V, d) \xrightarrow{\simeq} (\Delta W, d)$$

then, g is an isomorphism

Idea for the proof:
from the lifting problem

$$\begin{array}{ccc} & \xrightarrow{\neq} & (\Delta V, d) \\ & & \simeq \downarrow g \\ (\Delta W, d) & \xrightarrow{i} & (\Delta W, d) \end{array}$$

by the homotopy lifting lemma it follows that $g \circ f \simeq \text{id}$
and then try to find that actually $g \circ f = \text{id}$

the uniqueness of the model

Suppose $g: (\Delta V, d_V) \xrightarrow{\cong} (A, d_A)$ is a minimal Sullivan model of the commutative cochain algebra (A, d_A) .

then, $(\Delta V, d_V)$ is unique up to isomorphism.

Proof: Suppose $f: (\Delta W, d_W) \xrightarrow{\cong} (A, d_A)$ is a second minimal model. Form the lifting problem

$$\begin{array}{ccc} & \tilde{f} & (\Delta V, d_V) \\ & \nearrow & \downarrow g \\ (\Delta W, d_W) & \xrightarrow{f} & (A, d_A) \end{array}$$

Due to the lifting lemma, we have the lift $\tilde{f}$ with $g \circ \tilde{f} \cong f$.

Due to Prop. P2, we derive that $H(g) \circ H(\tilde{f}) = H(f)$ and, as

$H(g)$ and $H(f)$ are isomorphisms, so is $H(\tilde{f})$, i.e. $\tilde{f}$ is a quasi-isomorphism

By the proposition of the previous slide, we have that $\tilde{f}$ is an isomorphism.

Sullivan representative

Let $h: (A, d_A) \longrightarrow (B, d_B)$ denote a morphism of connected commutative cochain algebras.

As we know, both algebras have their own minimal models

$$\begin{array}{ccc} (A, d_A) & \xrightarrow{h} & (B, d_B) \\ \uparrow \varphi_A & & \uparrow \varphi_B \\ (\Delta V, d_V) & \xrightarrow{\hat{h}} & (\Delta W, d_W) \end{array}$$

the morphism $\hat{h}$
is called a
Sullivan representative
of h

Due to the lifting lemma and the homotopy lifting lemma,

there is a lift unique up to homotopy of $(h \circ \varphi_A)$ along φ_B

to a morphism $\hat{h} = \widetilde{h \circ \varphi_A} : (\Delta V, d_V) \longrightarrow (\Delta W, d_W)$

which makes the diagram homotopy commute,

$$\text{i.e. } h \circ \varphi_A \simeq \varphi_B \circ \hat{h}$$

Combining the results

Suppose we are given a chain of weak equivalences of simply-connected commutative cochain algebras

$$\begin{array}{ccccccc} (A, d_A) & \xleftarrow{\simeq} & (Z_1, d_{Z_1}) & \xrightarrow{\simeq} & \cdots & \xleftarrow{\simeq} & (Z_k, d_{Z_k}) \xrightarrow{\simeq} (B, d_B) \\ \uparrow \varphi_A \simeq & & \uparrow \varphi_{Z_1} \simeq & & \uparrow \simeq & & \uparrow \varphi_{Z_k} \simeq & & \uparrow \varphi_B \simeq \\ (\Delta V, d_V) & \xleftarrow{\simeq} & (\Delta M_1, d_{M_1}) & \xrightarrow{\simeq} & \cdots & \xleftarrow{\simeq} & (\Delta M_k, d_{M_k}) \xrightarrow{\simeq} & (\Delta W, d_W) \end{array}$$

We can also consider their minimal models

Due to the existence of Sullivan representatives, we obtain this diagram which is homotopy commutative.

Since the morphisms in the upper line as well as the vertical morphisms are quasi-isomorphisms, so are the Sullivan representatives. Consequently, these are isomorphisms.

Corollary

Weakly equivalent simply-connected commutative cochain algebras have isomorphic minimal Sullivan models.

$$\left\{ \begin{array}{l} \text{weak equivalence} \\ \text{classes of connected} \\ \text{cochain algebras} \end{array} \right\} \xleftrightarrow{1:1} \left\{ \begin{array}{l} \text{isomorphism classes of} \\ \text{minimal Sullivan models} \end{array} \right\}$$

Thank you

for your attention

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